KOSTIA BEIDAR'S CONTRIBUTIONS TO MODULE AND RING THEORY

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ABSTRACT. At the beginning of his mathematical career Kostia Beidar was working on rings with polynomial identities and primeness conditions for rings. By Posner's theorem the two-sided quotient ring of a prime PI-ring is a finite matrix ring over some field. This result was extended by Martindale to rings with generalised polynomial identities by the construction of the central closure of a prime ring. Kostia was working extensively in this setting and made crucial contributions to the understanding of the theory. While his contribution to general PI theory will be outlined elsewhere we want to sketch here his work on prime rings and the resulting study of (strongly) prime modules. An account on his papers on Hopf algebras is given and attention is drawn to some more recent constructions which grew out from Kostia's basic contributions to this field.

CONTENTS

1. RINGS AND RINGS OF QUOTIENTS

1.1. **Finite automorphism groups.** The two early papers [7] and [6] by Kostia are dealing with automorphism groups of algebras. Let G be a finite group acting on an associative ring A with unity, with A^G the fixed ring of this action. An element $\gamma \in A$ is called an element of trace one if

$$\mathrm{tr}_G(\gamma) = \sum_{g \in G} g(\gamma) = 1.$$

In [7] it is proved that if A has an element of trace one, then A inherits each of the following two properties from A^G:

(i) every quotient ring by a primitive ideal is Artinian;

The first author was partially supported by the Centro de Matemática da Universidade do Porto, financed by FCT (Portugal) through the programmes POCTI and POSI, with national and European Community structural funds.

(ii) the ring is a PI-ring.

The second property was shown later by Kostia and B.Torrecillas in [46] to hold also for finite dimensional Hopf algebras H with cocommutative coradical acting on an unital algebra A such that $t \cdot \gamma = 1$ for some $\gamma \in A$ and a left integral $t \in H$.

Denote by $N(A)$ the upper nil-radical of the algebra A. Herstein's conjecture asks whether $A/N(A)$ is commutative, provided that A^G lies in the centre of A and G is a cyclic group of prime order. Kostia gave an affirmative answer in [7] under the additional assumption that A^G is semiprime or that A has an element $\gamma \in A$ such that $\mathrm{tr}_G(\gamma)$ is central in A and not a zero divisor of A.

Under the global assumption that G is a finite cyclic group and A^G lies in the centre of A, he had shown already in [6, Theorem 4] that the existence of an element $\gamma \in A$ where $\mathrm{tr}_G(\gamma)$ is central and not a zero divisor in A implies the commutativity of $A/\mathrm{rad}\, A$, where $\mathrm{rad}\, A$ is the classical radical of A. Moreover [6, Theorem 2] shows that the action of G on A can be extended to an action of G on $Q(A)$, the maximal right ring of quotients of A, provided A contains a central element of trace one and satisfies $\mathrm{rad}\, A = 0$. In this situation one also has $Q(A)^G = Q(A^G)$.

For convenience recall that the *maximal (or complete) right ring of quotients* of a ring A is defined as the ring

$$Q(A) = \{b \in E(A) \mid \text{for any } f \in \mathrm{End}\, E(A), f(A) = 0 \text{ implies } f(b) = 0\},$$

where $E(A)$ denotes the injective hull of A as right A-module.

1.2. **Non-degenerated alternative rings.** Kostia has written several joint papers on nonassociative rings. Recall that a ring A is said to be *alternative* if $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in A$. An alternative ring A is *non-degenerate* if $xAx \neq 0$ for any non-zero element $x \in A$. In [22] it is proved:

Theorem. *For a non-degenerate alternative ring A, the following are equivalent:*

(a) *A is prime, that is $IJ \neq 0$ for any two nonzero ideals $I, J \subset A$;*
(b) *$(aA)b \neq 0$ for any nonzero $a, b \in A$;*
(c) *$a(Ab) \neq 0$ for any nonzero $a, b \in A$.*

A similar result is obtained for Jordan rings in [22, Theorem 2] where non-degeneracy is defined by the Jordan triple product of elements.

The study of non-degenerate alternative rings was continued in [23]. There the construction of *nearly classical localization* is given and the structure of non-degenerate alternative algebras is described. Purely alternative (that is, nonassociative) alternative rings are called *generalised Cayley-Dickson rings.*

Theorem. ([23, Corollary 2.15]) *Let A be a non-degenerate alternative algebra. Then A is a subdirect product of semiprime associative algebras and of generalised Cayley-Dickson rings.*

That Cayley-Dickson rings play a dominant role for purely alternative rings was shown in [23, Theorem 2.16]:

Theorem. *Let A be a non-degenerate alternative algebra over a commutative associative noetherian ring R with unit. Then A is either an associative algebra or A contains a subalgebra which is a Cayley Dickson ring.*

In Kostia's paper [32] it is proved that the centre of a non-degenerate purely alternative algebra A contains a dense ideal I such that for any nonzero $t \in I$ the classical localization A_t of the algebra A with respect to t is a Cayley-Dickson algebra over its centre. This is used to show that the classical ring of quotients of an alternative PI-algebra is a PI-algebra and some of the results are applied to the description of von Neumann regular alternative algebras. Let us mention that purely alternative prime algebras behave similar to prime PI-algebras. For example, for both types of rings the nonzero ideals have nonzero intersections with the centre.

1.3. **Orthogonal completeness.** Although it is generally useful to study semiprime rings by reducing the questions to prime rings, this approach frequently presents some difficulties. For example, it is well known that every polynomial identity of a prime ring R is also an identity of its maximal right ring of quotients $Q(R)$. However, when trying to prove a similar statement for semiprime rings, the direct reduction to prime rings is not so easy, since in general there is no homomorphism $Q(R) \to Q(R/P)$, where P is a prime ideal of the ring R.

Some of the problems arising in this context can be overcome by the theory of *orthogonal completeness* which was investigated and developed in a series of papers by Kostia mainly in cooperation with A.B. Mikhalev [18, 20, 21, 19] and a nice overview of these results is given in [39].

Recall that a unital ring B is called *Boolean* if every element of B is an idempotent. An algebra A over a Boolean ring B is said to be *orthogonally complete* if for any $a \in A$ the ideal

$$r(B; a) = \{b \in B \mid ab = 0\}$$

is principal, and if for any dense orthogonal subset $E \subseteq B$ and for any family of elements $S = \{s_e | e \in E\} \subseteq A$ there exists an element $a \in A$ such that $ea = es_e$ for all $e \in E$.

One way of obtaining an orthogonal completion of an algebra A over an associative semiprime commutative ring K is by *almost classical localization.* For this, let $\mathcal{F}$ be the filter of dense ideals of K, and assume that A is $\mathcal{F}$-torsion free. It turns out that $A_{\mathcal{F}}$ is an orthogonally complete $K_{\mathcal{F}}$-algebra over the Boolean ring B of idempotents of $K_{\mathcal{F}}$. The orthogonal completion of A is then the intersection of all orthogonally complete subalgebras of $A_{\mathcal{F}}$ containing A.

The method of orthogonal completeness has three components:

(1) Constructions and descriptions of orthogonal completions,
(2) sufficient conditions for the primeness of Pierce stalks of orthogonally complete rings, and
(3) metatheorems which allow one to lift structure theorems to the orthogonally complete rings from their Pierce stalks.

The theory was applied by the authors in various situations, for example for the structure of nondegenerate alternative algebras and the structure of semiprime rings with bounded indices of nilpotent elements. We mention two typical results, [11, Theorem 2 and 3].

Theorem. *Let $Q = Q(A)$ be the maximal right ring of quotients of the prime ring A. Then the following conditions are equivalent:*

(a) *A is a right Goldie ring;*
(b) *there exists some $n \in \mathbb{N}$ such that $a^n \in a^{n+1}Q$ for all $a \in A$;*
(c) *the indices of the nilpotent elements of A are bounded, and for each $a \in A$ there exists $n = n(a) \in \mathbb{N}$ such that $a^n \in a^{n+1}Q$*

Theorem. *Let $Q = Q(A)$ be the maximal right ring of quotients of the semiprime ring A. Then the following are equivalent:*

(a) *Q is the direct sum of finitely many matrix rings over strongly regular right self-injective rings;*
(b) *there exists some $n \in \mathbb{N}$ such that $a^n \in a^{n+1}Q$ for all $a \in A$.*

1.4. **The central closure of a prime ring.** Let R be prime ring and let $\mathbb{U} = \{U \mid U \text{ is a nonzero ideal in } R\}$. For any non-zero $U, V \in \mathbb{U}$, consider homomorphisms $f : U_R \to R_R$ and $g : V_R \to R_R$ and define f to be equivalent to g if $f|_{U\cap V} = g|_{U\cap V}$. This defines an equivalence relations on the set of all such morphisms. With obvious addition and multiplication the set of the equivalence classes form a ring Q, the *Martindale ring of quotients* of A. The centre C of Q is a field and is called the *extended centroid* of R. Then $S = RC \subset Q$ is a prime ring which is called the *central closure* of R.

In [58, Theorem 3] it is shown that S satisfies a generalised polynomial identity if and only if S contains a minimal right ideal eS, $e^2 = e \in S$, and eSe is a finite dimensional division algebra over C.

Let R be a semiprime ring. As outlined in [59], the above construction can be repeated by replacing the set $\mathbb{U}$ of all ideals by all essential ideals of R. It was shown by Kharchenko that in this case the extended centroid C is a regular ring and Kostia proved (in [2, Theorem 1]) that C is a self-injective ring. Moreover the central closure $S = RC$ is a semiprime ring. As shown in [54] associativity of the ring R is not needed for the construction of the central closure. This will also follow from the module theoretic constructions considered in 3.2.

When is the central closure simple? One of the questions of central localization is when the central closure $S = RC$ is a simple ring. If R be a prime ring, then S is a prime ring with the centre C (extended centroid) being a field. If S satisfies a polynomial identity, then the intersection of any ideal with the centre C is non-zero and hence contains an invertible element. Thus S is a simple ring. Moreover, PI theory tells us that S has finite dimension as C-vector space.

The question arises which property of the ring R (other than PI) implies that S is a simple ring. This may also be expressed by properties of the (R, R)-bimodule R and this will be done in 3.1. To prepare this some new notions in module theory are needed.

2. Strongly prime and semiprime modules

To provide the techniques to deal with the questions asked above recall that for any R-module M, the full subcategory of $R-$Mod whose objects are submodules of M-generated modules is denoted by $\sigma[M]$. This is a Grothendieck category and every object has an (M-)injective hull in $\sigma[M]$.

Let $\widehat{M}$ denote the M-injective hull of $M \in \sigma[M]$. The class

$$\{X \in \sigma[M] \,|\, \mathrm{Hom}_R(X, \widehat{M}) = 0\}$$

is a torsion class and induces a torsion theory in $\sigma[M]$.

We write $U \trianglelefteq M$ to indicate that U is an essential submodule of M. The module M is called *polyform* provided $\mathrm{Hom}_R(M/U, \widehat{M}) = 0$ for every $U \trianglelefteq M$. Notice that a ring R is left polyform if and only if its left singular submodule is zero.

2.1. **Bimodule properties of polyform modules.**

Of particular interest is the bimodule structure of polyform modules. Motivated by the properties observed for nonsingular and semiprime rings and Kostia's experience with idempotents, the following is shown in [36, 3.3].

Theorem. *Let M be a polyform R-module, $\widehat{M}$ its M-injective hull and $T = \mathrm{End}_R(\widehat{M})$. Denote by C the center of T (i.e., the endomorphism ring of $\widehat{M}$ as an (R, T)-bimodule). Then:*

1. *Every essential (R, T)-submodule of $\widehat{M}$ is essential as an R-submodule.*
2. *$\widehat{M}$ is self-injective and polyform as an (R, T)-bimodule.*
   C is a regular self-injective ring.
3. *For every submodule (subset) $K \subset \widehat{M}$, there exists an idempotent $\varepsilon(K) \in C$, such that $An_C(K) = (1 - \varepsilon(K))C$.*
4. *If $K \trianglelefteq L \subset \widehat{M}$, then $\varepsilon(K) = \varepsilon(L)$.*
5. *Every finitely generated C-submodule of $\widehat{M}$ is C-injective.*
6. *If $\widehat{M}$ is a finitely generated (R, T)-module, $\widehat{M}$ is a generator in C-Mod.*

These observations lead to the definition and properties of the

Theorem (Idempotent closure of polyform modules)**.** *Let M be an R-module, $T = \mathrm{End}_R(\widehat{M})$ and B the Boolean ring of all central idempotents of T. We call $\widetilde{M} = MB$ the* idempotent closure of M.

Then for every $a \in \widetilde{M}$, there exist $m_1, \ldots, m_k \in M$ and pairwise orthogonal $c_1, \ldots, c_k \in B$ such that $a = \sum_{i=1}^k m_i c_i$.

If M is a polyform module, there exist pairwise orthogonal idempotents $e_1, \ldots, e_k \in B$ such that

(1) $a = \sum_{i=1}^k m_i e_i$;
(2) $e_i = \varepsilon(m_i) e_i \quad$ *for* $i = 1, \ldots, k$;
(3) $\varepsilon(a) = \sum_{i=1}^k e_i$.

2.2. **Strongly semiprime modules.** Again this point of view can be extended to semiprimeness and here Kostia came in with substantial contributions in [36], [37], and [38]. A module M is called *strongly semiprime*, for short *SSP*, if its M-injective hull $\widehat{M}$ is semisimple as an (R, T)-bimodule where $T = \mathrm{End}_R(\widehat{M})$ (see [37, 4.5]).

Theorem. *Let M be an R-module.*

(1) *Assume M has an essential socle and for every $N \trianglelefteq M$, $M \in \sigma[N]$. Then M is semisimple.*
(2) *M is semisimple if and only if every module in $\sigma[M]$ is SSP.*

In general SSP modules need not be polyform. However we have the following theorem:

Theorem (Projective strongly semiprime modules)**.** *Let M be projective in $\sigma[M]$ and $T = \mathrm{End}_R(\widehat{M})$. Then the following are equivalent:*

(a) *M is an SSP-module;*
(b) *M is polyform and for any $N \trianglelefteq M$, $M \in \sigma[N]$.*

Applied to the case $M = R$ we get a characterization of left SSP rings ([37, 8.2]):

Theorem. *For a ring R let $Q = Q(R)$ denote the maximal left ring of quotients. Then the following are equivalent:*

(a) *R is left SSP;*
(b) *for every essential left ideal $N \subset R$, $R \in \sigma[N]$;*
(c) *every $N \trianglelefteq {}_R R$ contains a finite subset X with $An_R(X) = 0$;*
(d) *R is semiprime and every left ideal $I \subset R$ contains a finite subset $X \subset I$ with $An_R(X) = An_R(I)$;*
(e) *Q is a semisimple (R, Q)-module.*

If R satisfies these conditions, then Q is left self-injective, von Neumann regular, and a finite product of simple rings. Left ideals with property (c) in 2.2 are also called *insulated.* So the rings described here are exactly the *left strongly semiprime rings* as considered by Handelman [53].

2.3. **Strongly prime modules.** The R-module M is called *strongly prime* if every nonzero submodule $K \subset M$ is a subgenerator in $\sigma[M]$, that is, $M \in \sigma[K]$.

Theorem. *For an R-module M with $T = \mathrm{End}_R(\widehat{M})$, the following are equivalent:*

(a) *M is strongly prime;*

(b) *M is SSP and $\widehat{M}$ is a uniform (R,T)-bimodule.*

(c) *$\widehat{M}$ is a simple (R,T)-bimodule;*

(d) *$\widehat{M}$ has no fully invariant submodule.*

In particular, for a uniform R-module M, the conditions strongly prime and SSP are equivalent.

3. The bimodule structure of an algebra

As pointed out earlier the motivation for some of the notions introduced for modules was to understand the bimodule structure of an algebra. This will be outlined in this section.

3.1. **Bimodule structure of an algebra.** For any algebra (or ring) A and $a \in A$, the left and right multiplications

$$L_a : A \to A,\ x \mapsto ax, \quad A_a : A \to A,\ x \mapsto xa,$$

are $\mathbb{Z}$-linear maps and the *multiplication algebra* $M(A)$ of A is defined as the subring of $\mathrm{End}_{\mathbb{Z}}(A)$ generated by all L_a, A_a, $a \in A$ and the identity map of A. Notice that we do not require A to be associative nor to have a unit.

Then A is a left module over $M(A)$ and $\mathrm{End}_{M(A)}(A)$ is called the *centroid* of A. If A has a unit, then this is isomorphic to the center $Z(A)$ of A.

In general A is not a generator in $M(A)-\mathrm{Mod}$ and to relate properties of A with properties of $M(A)$-modules one has to restrict to the full subcategory $\sigma[A]$ of $M(A)-\mathrm{Mod}$ whose elements N are subgenerated by A, that is, N is a submodule of an A-generated $M(A)$-module. If A has a unit then an $M(A)$-module N is A-generated if and only if it is generated by its central elements $\{m \in N \,|\, am = ma \text{ for all } a \in A\}$.

Notice that in the category $\sigma[A]$, every object has an injective hull. In particular, the selfinjective hull $\widehat{A}$ of the $M(A)$-module A is injective in $\sigma[A]$ and is an A-generated $M(A)$-module, that is, $\widehat{A} = A\mathrm{Hom}_{M(A)}(A, \widehat{A})$.

3.2. **Central closure of semiprime algebras.** For the construction of the maximal left ring of quotients of a semiprime associative ring it is of interest if the ring is left non-singular. Although every semiprime commutative ring is non-singular, a noncommutative semiprime ring need not be non-singular as a left (or right) module over itself. However, any semiprime ring A is non-singular in the category $\sigma[A]$ as an (A,A)-bimodule and this makes it possible to construct a quotient ring for any semiprime ring.

Theorem. ([37, 9.1]). *Let A be a semiprime algebra with A-injective hull $\widehat{A}$ as $M(A)$-module and $T := \mathrm{End}_{M(A)}(\widehat{A})$ (the extended centroid). Then:*

(1) *A is a polyform $M(A)$-module.*
(2) *T is a commutative, regular, and self-injective ring.*
(3) *$\widehat{A} = AT$ is a semiprime ring with respect to the multiplication*

$$(as) \cdot (bt) := (ab)st, \quad \text{for } a, b \in A, \ s, t \in T,$$

and linear extension.

$\widehat{A}$ is called the *central closure* of A.

In the given situation the idempotent closure of a polyform module yields a ring extension.

Theorem (Idempotent closure of semiprime algebras)**.** *Let A be a semiprime R-algebra, $T = \mathrm{End}_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T. The idempotent closure of A as an $M(A)$-module, $\widetilde{A} = AB$ (see 2.1), is an R-algebra and*

(1) *for any $a \in \widetilde{A}$, there exist $a_1, \ldots, a_k \in A$ and pairwise orthogonal $e_1, \ldots, e_k \in B$, such that*
   (i) $a = \sum_{i=1}^k a_i e_i$,
   (ii) $e_i = \varepsilon(a_i) e_i$, *for* $i = 1, \ldots, k$, *and*
   (iii) $\varepsilon(a) = \sum_{i=1}^k e_i$.
(2) *For every prime ideal $K \subset \widetilde{A}$, $P = K \cap A$ is a prime ideal in A and*

$$\widetilde{A}/K = (A + K)/K \simeq A/P.$$

The set $x = \{e \in B | \widetilde{A}e \subset K\}$ is a maximal ideal in B and $K = PB + \widetilde{A}x$.
(3) *For any prime ideal $P \subset A$, there exists a prime ideal $K \subset \widetilde{A}$ with $K \cap A = P$.*

Of course for any prime algebra the central closure can be constructed as in 3.2 and we obtain special properties.

Theorem (Central closure or prime algebras)**.** *Let A be a prime algebra with A-injective hull $\widehat{A}$ as $M(A)$-module and $T := \mathrm{End}_{M(A)}(\widehat{A})$. Then:*

(1) *T is a field.*
(2) *$\widehat{A} = AT$ is a prime ring whose center is a field.*
(3) *$\widehat{A}$ is a simple ring if and only if A is strongly prime as an $M(A)$-module.*

Recall that a ring A is an *Azumaya ring* if A is a generator in the category $\sigma[A]$ of $M(A)$-modules. We mention the result [36, 9.9] showing some relations with this type of algebras.

Theorem. *Let A be a semiprime ring with unit, $T = \mathrm{End}_{M(A)}(\widehat{A})$ where $\widehat{A}$ is the central closure of A (see 3.2). Then the following are equivalent:*

(a) *$\widehat{A}$ is an Azumaya ring;*
(b) *$\widehat{A}$ is a biregular ring and $\widehat{A}$ is a projective module in $\sigma_{M(\widehat{A})}[\widehat{A}]$;*
(c) *the $M(\widehat{A})$-module $\widehat{A}$ is a generator in $\sigma_{M(\widehat{A})}[\widehat{A}]$;*
(d) *$M(\widehat{A})$ is a dense subring in $\mathrm{End}_T(\widehat{A})$.*

3.3. **Central closure for Hopf module algebras.** Given an algebra A with an action of a Hopf algebra H, we can similarly construct a kind of central closure for A to which we can extend the action of H. More generally, the above construction works for any extension $A \subseteq B$ of unital rings such that there exists a ring homomorphism $\varphi : B \to \mathrm{End}(A)$ with $M(A) \subseteq \mathrm{Im}(\varphi)$. By [57] we have the following theorem:

Theorem. *Let $A \subseteq B$ as above and denote by $\widehat{A}$ the self-injective hull of A as a B-module. Assume that A is B-semiprime, i.e., has no non-zero B-stable nilpotent ideal. Then*

(1) *A is a polyform B-module and $A^B = \mathrm{End}_B(A)$ is a commutative reduced ring.*
(2) *$\widehat{A}^B = \mathrm{End}_B(\widehat{A})$ is a commutative von Neumann regular self-injective ring, which is a field if and only if A is B-prime, i.e. the product of two non-zero B-stable ideals of A is non-zero.*
(3) *$\widehat{A} = A\widehat{A}^B$ is a semiprime ring with respect to*
$$(as) \cdot (bt) := (ab)st, \qquad \text{for } a, b \in A,\ s, t \in \widehat{A}^B,$$
and linear extension.

Since $\widehat{A}$ is a B-module, the action of B on A extends to an action of B on $\widehat{A}$. As an application to Hopf algebra action one defines a new multiplication on the tensor product $B = A^e \otimes H$, where $A^e = A \otimes A^{op}$ is the enveloping algebra and H is a Hopf algebra acting on A, such that $A \subseteq B$ is an extension as described above. The construction of $\widehat{A}$ yields a new H-module algebra which coincides with the central closure constructed by Matczuk.

3.4. **Strongly semiprime algebras.** Under some non-degeneracy condition strongly semiprime algebras are semiprime and thus the central closure is defined yielding [37, 9.4].

Theorem. *Let A be a ring which is not annihilated by any non-zero ideal and $T = \mathrm{End}_{M(A)}(\widehat{A})$. Then the following conditions are equivalent:*

(a) *A is an SSP $M(A)$-module;*
(b) *A is a semiprime algebra and for every essential ideal $U \subset A$, $A \in \sigma_{M(A)}[U]$;*
(c) *A is semiprime and the central closure $\widehat{A}$ is a direct sum of simple ideals.*

If A is associative, then (a)-(c) are equivalent to:

(d) *A is semiprime and for every ideal $I \subset A$, $A/An_A(I) \in \sigma_{M(A)}[I]$.*

4. Hopf algebras and quantum Yang-Baxter equation

In two papers Kostia, in cooperation with A. Stolin and Y. Fong, proved a couple of interesting results on Frobenius algebras over commutative rings that they could apply successfully to Hopf algebras over commutative rings.

4.1. **Frobenius Algebras and quantum Yang-Baxter equation.** An algebra A over a commutative ring K is called *Frobenius* if it is a finitely generated projective K-module and there exists $\phi \in A^* = \mathrm{Hom}(A, K)$ such that the map $\psi : A \to A^*$ with $\psi(x)(y) = \phi(yx)$ for all $x, y \in A$ is an isomorphism of K-modules. Since A is finitely generated and projective as K-module, it has a dual basis $\{e_i, f^i\}_{1\leq i\leq n}$, i.e. $e_i \in A$ and $f^i \in A^*$ such that

$$x = \sum_{i=1}^{n} f^i(x)e_i.$$

Then given any $x \in A$, there exists a unique $x' \in A$ such that

$$\phi(yx') = \psi(x')(y) = \phi(xy)$$

for all $y \in A$. This defines an automorphism $\alpha : A \to A$ with $\alpha(x) = x'$, called the *Nakayama automorphism* of A.

Let $Q = \sum a_i \otimes b_i \in A \otimes A$ be an element of the tensor product. We will use the following notations for elements in the 3-fold tensor of A:

$$Q^{12} = \sum a_i \otimes b_i \otimes 1 \in A^{\otimes 3},$$
$$Q^{13} = \sum a_i \otimes 1 \otimes b_i \in A^{\otimes 3},$$
$$Q^{23} = \sum 1 \otimes a_i \otimes b_i \in A^{\otimes 3}.$$

If $R \in \mathrm{End}_K(A \otimes A)$ then we also use the notation R^{12}, R^{13}, R^{23} to denote the endomorphisms of the 3-fold tensor $A^{\otimes 3}$ acting on the components indicated by the superscripts, i.e. $R^{12} = R \otimes id$, $R^{23} = id \otimes R$ and $R^{13}(x\otimes y\otimes z) = (id\otimes T)R(x\otimes z)\otimes y$ where T is the twist map $T(a\otimes b) = b\otimes a$.

Theorem ([41, Theorem 3.4])**.** *Let A be a Frobenius algebra over K with Frobenius homomorphism ϕ, Nakayama automorphism α, and dual bases (e_i, f^i) and $e^i := \psi^{-1}(f^i)$. Set $Q = \sum_{i=1}^{n} e_i \otimes e^i \in A \otimes_K A$ and define $T \in \mathrm{End}_K(A \otimes_K A)$ by $T(a \otimes b) = b \otimes a$. Then Q satisfies the braid relation*

$$Q^{12}Q^{23}Q^{12} = Q^{23}Q^{12}Q^{23}$$

and $R = QT \in \mathrm{End}_K(A \otimes_K A)$ satisfies the quantum Yang-Baxter equation (QYBE)

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

Moreover $\mathcal{O}(A) = \{P \in A \otimes_K A \mid (1 \otimes a)P = P(a \otimes 1)\}$ is a free rank one left (right) A-submodule of $A \otimes_K A$ with basis $\{Q\}$.

Let $Z(A)$ denote the center of the Frobenius K-algebra A and define $\Phi : A \to Z(A)$ as $\Phi(x) = \sum_{i=1}^{n} e^i x e_i$ for all $a \in A$. Set $u = \Phi(1)$.

Theorem ([41, Theorem 4.2]). *Consider the following conditions:*

(1) *u is invertible in A;*
(2) *A is a Frobenius $Z(A)$-algebra with Frobenius homomorphism Φ;*
(3) *$\Phi(x) = 1$ for some $x \in A$;*
(4) *A is a separable K-algebra.*

Then the implications $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ hold.

In particular, if A satisfies one of conditions $(ii), (iii)$ or (iv) then A is a separable K-algebra with separability idempotent $\sum e_i \otimes e^i x$, where x is the element of condition (iii).

4.2. **Hopf algebras as Frobenius algebras.** Let H be a Hopf algebra over a commutative ring K with antipode S, comultiplication Δ, and counit ε. B. Pareigis showed that every Hopf algebra H that is finitely generated projective over a commutative ring K with trivial Picard group, is a Frobenius K-algebra with Frobenius homomorphism $\phi \in H^*$ such that ϕ satisfies

$$\sum_{(h)} h_1 \phi(h_2) = \phi(h) \cdot 1$$

for all $h \in H$, where $\sum_{(h)} h_1 \otimes h_2 = \Delta(h)$ is the comultiplication of h in Sweedler's notation. The latter condition on ϕ says that ϕ is an H-colinear map.

Recall that a left integral in H is an element $t \in H$ such that $ht = \varepsilon(h)t$ for all $h \in H$, and that H^* is also a Hopf algebra over K and denote its counit by π.

Theorem ([42, Theorem 3.2]). *Let H be a Hopf algebra over K which is Frobenius with a Frobenius homomorphism ϕ which is H-colinear. Then*

(1) *$N = \sum_{i=1}^{n} \varepsilon(e_i) e^i$ is a left integral in H and a left norm, i.e. $\phi(xN) = \varepsilon(x)$ for all $x \in H$.*
(2) *ϕ generates the submodule of left integrals in H^* and*
$$\mathrm{Tr}(S^2) = \varepsilon(N)\pi(\phi).$$
(3) *Given any left integral $l \in H$,*
$$R = (S^{-1} \otimes 1)\Delta(l)T \in \mathrm{End}(H \otimes_K H)$$
is a solution of QYBE.

The trace formula for the square of the antipode, which in the case that K is a field is due to *R. G. Larson* and *D. E. Radford* and was used in an essential way in their proof of a conjecture of Kaplansky (see [55, 56]).

Applying the last Theorem and the characterisation of separable Frobenius algebras yields the following corollaries which generalise known results for Hopf algebras over fields:

Corollary ([42, Corollaries 3.4 and 3.5]). *Let H be a Hopf algebra over a commutative ring K such that H is finitely generated projective as K-module. Denote the antipode of H by S.*

(1) *H is a separable K-algebra if and only if H has a left integral l such that $\varepsilon(l)$ is an invertible element of K.*
(2) *H and H^* are separable K-algebras if and only if* $\mathrm{Tr}(S^2)$ *is an invertible element of K.*

A Hopf algebra H is called *involutory* if its antipode S is an involution, i.e., $S^2 = id$. As an extension of a theorem by Larson we have now:

Theorem. *The following conditions are equivalent for a Frobenius algebra H over an algebraically closed field K.*
(a) *$u = \sum_{i=1}^{n} e_i e^i$ is invertible;*
(b) *H is a separable K-algebra and the characteristic of K does not divide dimensions of simple H-modules.*

In particular, condition (a) is fulfilled if H is an involutory semisimple K-Hopf algebra.

If H is unimodular, i.e., the submodules of left integrals and of right integrals coincide, and finitely generated and projective as a K-module, then S acts as the identity on the submodule of integrals and $S^4 = id$ holds provided H^* is also unimodular.

5. Structure of matrix rings

Let n be a positive integer and R a ring, and let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

5.1. **CS matrix rings over local rings.** In the papers [47, 48, 49, 50] with various coauthors Kostia made some contributions to the structure theory of CS modules and rings. A module M is called *CS* or *extending* provided every submodule of M is essential in a direct summand. M is said to be a *tight* module if every finitely generated submodule of the self-injective hull of M embeds in M. A ring R is called a *right CS-ring* if R is CS as a right R-module, and R is *right tight* provided it is tight as a right module.

An open problem is to find necessary and sufficient conditions for direct sums of CS-modules to be CS. In [48] it is shown that if M is non-M-singular and CS, then M is M-tight and $\mathrm{End}(M_R)$ is right PP, and the converse also holds if M is furthermore a self-generator.

This result is applied to give necessary and sufficient conditions for R^n to be CS as a right R-module (equivalently, the $n \times n$ matrix ring $M_n(R)$ is a right CS-ring), where R is either a reduced ring or a ring with no infinite set of nonzero orthogonal idempotents. In particular, the open problem of characterizing a domain R such that R^2 is CS as a right R-module is solved; it is proved that such a domain is precisely a two-sided Ore domain and is two-sided 2-hereditary. Another result in this paper is:

Theorem. *For a von Neumann regular ring R, the following are equivalent for $n > 1$:*

(a) *$M_n(R)$ is right weakly selfinjective;*
(b) *$M_n(R)$ is right $M_n(R)$-tight;*
(c) *$M_n(R)$ is a right CS-ring;*
(d) *R is right selfinjective.*

In [47] a complete characterization of CS matrix rings $M_n(R)$, where $n > 1$, over local rings R is obtained:

Theorem. (1) *$M_n(R)$ is right CS if and only if R is right uniform and for every right ideal I of R and for every R-homomorphism $f\colon I \to R$ there exists $a \in R$ such that either $f = L_a$ or $L_a f = \mathrm{id}_I$, where L_a is the left multiplication by a and id_I is the identity map on I.*

(2) *If, in addition, the Jacobson radical of R coincides with the right singular ideal $\{r \in R \,|\, rE = 0$ for some essential right ideal of $R\}$, then $M_n(R)$ is a right CS-ring if and only if R is selfinjective.*

(3) *If R is a commutative Noetherian local ring, then $M_n(R)$ is a right CS-ring if and only if the classical two-sided quotient ring, $Q(R)$, is a local QF-ring such that for all $q \in Q(R)$ either $q \in R$ or q is invertible in Q and $q^{-1} \in R$.*

Applying the obtained results to group algebras, it is proved: If K is a field and G is a group (resp., nilpotent group) such that the group algebra KG is local (resp., semiperfect), then $M_n(KG)(n > 1)$ is a right CS-ring if and only if $\mathrm{char}(K) = p$ and G is a finite p-group (resp., finite group). This result was subsequently generalised by the same authors in [49].

Theorem. *Let K be a field and G be a group. Suppose that one of the following conditions is satisfied:*

(i) *G is a locally finite group;*
(ii) *the group algebra KG is semilocal and G is either a solvable group or a linear group.*

Then the following conditions are equivalent:

(a) *$M_n(KG)$ for $n > 1$ is a right CS-ring;*
(b) *$M_2(KG)$ is a right CS-ring;*
(c) *KG is right self-injective;*
(d) *G is a finite group.*

5.2. **Structure of right continuous right π-rings.** A right module M is called *π-injective* or *quasi-continuos* if $f(M) \subseteq M$ for every idempotent $f \in \mathrm{End}(E(M))$ where $E(M)$ is the injective hull of M. Quasi-continuos modules are in particular CS modules. A quasi-continuos module is called *continuos* if it is *direct injective*, i.e., if for every direct summand D of M every monomorphism $D \to M$ splits.

A ring R is called a right π-ring if every right ideal of R is π-injective. The structure of these rings was investigated in [50] leading to the following results.

For a positive integer n, let

(1) $D_1, D_2, \ldots, D_n$ be division rings,
(2) Δ be a right continuous right π-ring, all of whose idempotents are central, with essential ideal P such that Δ/P is a division ring and the right Δ-module Δ/P is not embeddable into Δ_Δ,
(3) V_i be a D_i-D_{i+1}-bimodule such that $\dim V_{iD_{i+1}} = 1$ for all $1 \leq i < n$
(4) V_n be a D_n-Δ-bimodule such that $V_n P = 0$ and $\dim V_{n\Delta/P} = 1$.

In this case, $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$ denotes the ring of $(n+1)$-by-$(n+1)$ matrices of the form

$$\begin{pmatrix} D_1 & V_1 & & & & \\ & D_2 & V_2 & & & \\ & & D_3 & V_3 & & \\ & & & \ddots & \ddots & \\ & & & & D_n & V_n \\ & & & & & \Delta \end{pmatrix}$$

with $V_i V_j = 0$ for all i, j.

The following result characterises right continuous right π-rings.

Theorem. *A ring R is a right continuous right π-ring if and only if R is the direct sum of finitely many rings of the form*

$$G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n),$$

finitely many indecomposable nonlocal right continuous right π-rings, and a right continuous right π-ring with all idempotents central.

5.3. **Uniform bounds of primeness in matrix rings.** A subset S of an associative ring R is a *uniform insulator* for R provided $aSb \neq 0$ for any nonzero $a, b \in R$. A ring R is called *uniformly strongly prime of bound* $m(R)$ if R has uniform insulators and the smallest of these has cardinality $m(R)$. The systematic studied of $m(R)$ was initiated by J. E. van den Berg who proved the following

Theorem. (1) *If F is an algebraically closed field, then* $m(\mathrm{M}_k(F)) = 2k - 1$.
(2) *Let F be a field and assume there exists a nonassociative division F-algebra of dimension k, then* $m(\mathrm{M}_k(F)) = k$.

He asked if the converse of (2) holds. In [51, Theorem 1.2] a positive answer to this question is given showing how it is related to the existence of nonassociative division algebras over F.

Theorem. *Let F be a field and k a positive integer k. Then:*

(1) $m(\mathrm{M}_k(F)) = 2k - 1$ *for all k if and only if F is algebraically closed;*
(2) $m(\mathrm{M}_k(F)) = k$ *if and only if there exists a nonassociative division F-algebra of dimension k.*

5.4. **Structure of rings with zero total.** The *total* was introduced by F. Kasch and Kostia was considering some questions arising from this notion. For two R-module M and N, $\mathrm{Rad}(M,N)$ is defined as the set of all $g \in \mathrm{Hom}(M,N)$ such that $1 - fg$ is an automorphism of M for all $f \in \mathrm{Hom}(N,M)$. Let $\Delta(M,N)$ denote the set of all $g \in \mathrm{Hom}(M,N)$ such that the kernel of g is an essential submodule of M. Finally, let $\mathrm{Tot}(M,N)$ be the set of all $g \in \mathrm{Hom}(M,N)$ such that for all $f \in \mathrm{Hom}(N,M)$, $fg \neq (fg)^2$ unless $fg = 0$.

In [45], a joint paper of Kostia with F. Kasch, conditions on R and the modules are studied so that all three ideals are equal. In the special case $M = R$, the total R is the subset

$$\mathrm{Tot}(R) = \{a \in R : aR \text{ does not contain nonzero idempotents}\}.$$

In [40], Kostia considers rings with $\mathrm{Tot}(R) = 0$ and obtained the following

Theorem ([40, Theorem 5]). *Let* $\mathrm{In}(a) = \min\{n \in \mathbb{N} \mid a^n = 0\}$ *for a nilpotent element* $a \in R$ *and* $\mathrm{In}(R) = \sup\{\mathrm{In}(a) \mid a \text{ nilpotent in } R)$. *For a ring* R *with* $\mathrm{In}(R) = n < \infty$, *the following are equivalent:*

(a) $\mathrm{Tot}(R) = 0$;
(b) *R contains an essential ideal I which is a direct sum of ideals $I_k = M_{n_k}(D_k)$, $k = 1, 2, \dots, t$, where*
   (1) $n_1 < n_2 < \cdots < n_t = n$.
   (2) *Each $M_{n_k}(D_k)$ is a matrix ring over a reduced ring D_k.*
   (3) *If $L \neq 0$ is a right ideal of D_k, then the set of all central idempotents of D_k belonging to L generates an essential ideal in L.*

This result yields interesting corollaries, e.g.

Corollary ([40, Corollary 6]). *The following statements are equivalent for a ring R with* $\mathrm{In}(R) = n < \infty$

(a) *R is a prime ring and* $\mathrm{Tot}(R) = 0$
(b) *$R \cong M_n(D)$, where D is a division ring;*

Corollary ([40, Corollary 8]). *Let R be a ring with* $\mathrm{Tot}(R) = 0$ *and* $\mathrm{In}(R) = n < \infty$. *Then the following statements hold:*

(1) *The maximal right ring of quotients Q of R equals the maximal left ring of quotients of R.*
(2) *Q is isomorphic to a finite direct sum of matrix rings over abelian regular left and right self-injective rings.*

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